Ballistic trajectory

The following example computes the two-dimensional ballistic trajectory of a projectile fired from a cannon, taking the velocity dependent drag slowing the projectile down into account.\(^1\) This drag causes the trajectory to deviate from a simple symmetric parabola as it will be steeper on its trailing half than on its leading half. The drag \(\delta(v)\) is assumed to be of the general form

\[
\delta(v) = rv^2 = r \left( \sqrt{x^2 + y^2} \right)^2
\]

which is a bit oversimplified but will suffice for the following. The general equations of motion of the projectile in this two-dimensional problem are

\[
\ddot{x} = -\frac{\delta(v)}{m} \cos(\varphi) \quad \text{and} \quad (1)
\]

\[
\ddot{y} = -g - \frac{\delta(v)}{m} \sin(\varphi) \quad (2)
\]

with \(g\) representing the acceleration of gravity, \(v\) denoting the projectile’s velocity, and \(m\) being its mass. Obviously, it is

\[
\cos(\varphi) = \frac{\dot{x}}{v} \quad \text{and} \quad \sin(\varphi) = \frac{\dot{y}}{v}.
\]

Setting the mass \(m := 1\) and rearranging (1) and (2), we get the

\(^1\) Cf. \([\text{Korn}, 1966, \text{pp. } 2-7 \text{ ff.}]\).
following set of differential equations:
\[
\ddot{x} = -\frac{\delta(v)}{v} \dot{x} \\
\ddot{y} = -g - \frac{\delta(v)}{v} \dot{y}
\]

The computer setup resulting from these equations is shown in figures 1 and 2. The upper and lower halves of the circuit are symmetric except for the input for the gravitational acceleration to the lower left integrator yielding \( \dot{y} \). The velocities \( \dot{x} \) and \( \dot{y} \) are fed to two multipliers yielding their respective squares which are then summed and square rooted to get \( v \) as we have \( \delta(v)/v = rv \).

The parameters \( \alpha_1 \) and \( \alpha_2 \) are scaling parameters that are set in order to get a suitably scaled picture on an oscilloscope operated in \( xy \)-mode. Table 1 shows the parameter set yielding the result shown in figure 3. The initial conditions satisfy
\[
\dot{x}_0 = \cos(\varphi_0) \quad \text{and} \\
\dot{y}_0 = \sin(\varphi_0)
\]
with \( \varphi \) denoting the elevation of the cannon. In this example \( \varphi_0 = 60^\circ \) has been chosen.

References

Figure 1: Computer setup for the simulation of a ballistic trajectory

<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
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<td>$\dot{y}_0$</td>
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<tr>
<td>$\alpha_2$</td>
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</tr>
</tbody>
</table>

Table 1: Parameter settings for the ballistic trajectory problem
Figure 2: Setup of the ballistic trajectory problem on an Analog Paradigm Model-1 analog computer

Figure 3: Ballistic trajectory