

Figure 1: Double pendulum

Simulating a double pendulum

This application note describes the simulation of a double pendulum as shown in figure 1 on an analog computer.¹

1 Equations of motion

The equations of motion will be derived by determining the Lagrangian $L = T - V$ with the two generalized coordinates φ_1 and φ_2 where T represents the total kinetic energy while V is the potential energy of the system. This potential energy is just

$$V = -g((m_1 + m_2)l_1 \cos(\varphi_1) + m_2 l_2 \cos(\varphi_2)) \quad (1)$$

with m_1 and m_2 representing the masses of the two bobs mounted on the tips of the two pendulum rods (which are assumed as being weightless). As always, g represents the gravitational acceleration.

¹This has also been done by [MAHRENHOLTZ, 1968, pp. 159–165] which served as an inspiration for this application note, especially the display circuit.

To derive the total kinetic energy the positions of the pendulum arm tips (x_1, y_1) and (x_2, y_2) are required:

$$\begin{aligned}x_1 &= l_1 \sin(\varphi_1) \\y_1 &= l_1 \cos(\varphi_1) \\x_2 &= l_1 \sin(\varphi_1) + l_2 \sin(\varphi_2) \\y_2 &= l_1 \cos(\varphi_1) + l_2 \cos(\varphi_2)\end{aligned}$$

The first derivatives of these with respect to time are

$$\begin{aligned}\dot{x}_1 &= \dot{\varphi}_1 l_1 \cos(\varphi_1), \\ \dot{y}_1 &= -\dot{\varphi}_1 l_1 \sin(\varphi_1), \\ \dot{x}_2 &= \dot{\varphi}_1 l_1 \cos(\varphi_1) + \dot{\varphi}_2 l_2 \cos(\varphi_2) \text{ and} \\ \dot{y}_2 &= -\dot{\varphi}_1 l_1 \sin(\varphi_1) - \dot{\varphi}_2 l_2 \sin \varphi_2.\end{aligned}$$

The kinetic energy of the system is then

$$T = \frac{1}{2} (m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_s(\dot{x}_2^2 + \dot{y}_2^2)) \quad (2)$$

with

$$\begin{aligned}\dot{x}_1^2 &= \dot{\varphi}_1^2 l_1^2 \cos^2(\varphi_1) \\ \dot{y}_1^2 &= \dot{\varphi}_1^2 l_1^2 \sin^2(\varphi_1) \\ \dot{x}_2^2 &= \dot{\varphi}_1^2 l_1^2 \cos^2(\varphi_1) + 2\dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \cos(\varphi_1) \cos(\varphi_2) + \dot{\varphi}_2^2 l_2^2 \cos^2(\varphi_2) \\ \dot{y}_2^2 &= \dot{\varphi}_1^2 l_1^2 \sin^2(\varphi_1) + 2\dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \sin(\varphi_1) \sin(\varphi_2) + \dot{\varphi}_2^2 l_2^2 \sin^2(\varphi_2)\end{aligned}$$

and thus

$$\begin{aligned}\dot{x}_1^2 + \dot{y}_1^2 &= \dot{\varphi}_1^2 l_1^2 \cos^2(\varphi_1) + \dot{\varphi}_1^2 l_1^2 \sin^2(\varphi_1) \\ &= \dot{\varphi}_1^2 l_1^2 (\cos^2(\varphi_1) + \sin^2(\varphi_1)) \\ &= \dot{\varphi}_1^2 l_1^2 \\ \dot{x}_2^2 + \dot{y}_2^2 &= \dot{\varphi}_1^2 l_1^2 (\cos^2(\varphi_1) + \sin^2(\varphi_1)) + \dot{\varphi}_2^2 l_2^2 (\cos^2(\varphi_2) + \sin^2(\varphi_2)) +\end{aligned} \quad (3)$$

$$\begin{aligned}
 & 2\dot{\varphi}_1\dot{\varphi}_2l_1l_2 (\cos(\varphi_1)\cos(\varphi_2) + \sin(\varphi_1)\sin(\varphi_2)) \\
 & = \dot{\varphi}_1^2l_1^2 + \dot{\varphi}_2^2l_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2l_1l_2 \cos(\varphi_1 - \varphi_2)
 \end{aligned} \tag{4}$$

The Lagrangian L results from equations (1) and (2) with (3) and (4) as

$$\begin{aligned}
 L = & \frac{1}{2}l_1^2\dot{\varphi}_1^2(m_1 + m_2) + \frac{1}{2}m_2\dot{\varphi}_2^2l_2^2 + m_2\dot{\varphi}_1\dot{\varphi}_2l_1l_2 \cos(\varphi_1 - \varphi_2) + \\
 & g(m_1 + m_2)l_1 \cos(\varphi_1) + gm_2l_2 \cos(\varphi_2).
 \end{aligned} \tag{5}$$

Now the EULER-LAGRANGE-equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_1} - \frac{\partial L}{\partial \varphi_1} = 0 \text{ and} \tag{6}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_2} - \frac{\partial L}{\partial \varphi_2} = 0 \tag{7}$$

have to be solved. The required (partial) derivatives are

$$\begin{aligned}
 \frac{\partial L}{\partial \varphi_1} & = -m_2\dot{\varphi}_1\dot{\varphi}_2l_1l_2 \sin(\varphi_1 - \varphi_2) - g(m_1 + m_2)l_1 \sin(\varphi_1) \\
 \frac{\partial L}{\partial \dot{\varphi}_1} & = l_1^2\dot{\varphi}_1(m_1 + m_2) + m_2\dot{\varphi}_2l_1l_2 \cos(\varphi_1 - \varphi_2) \\
 \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_1} & = l_1^2\ddot{\varphi}_1(m_1 + m_2) + m_2l_1l_2 [\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) - \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2)(\dot{\varphi}_1 - \dot{\varphi}_2)] \\
 \frac{\partial L}{\partial \varphi_2} & = m_2\dot{\varphi}_1\dot{\varphi}_2l_1l_2 \sin(\varphi_1 - \varphi_2) - gm_2l_2 \sin(\varphi_2) \\
 \frac{\partial L}{\partial \dot{\varphi}_2} & = m_2\dot{\varphi}_2l_2^2 + m_2\dot{\varphi}_1l_1l_2 \cos(\varphi_1 - \varphi_2) \\
 \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_2} & = m_2\ddot{\varphi}_2l_2^2 + m_2l_1l_2 [\ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) - \dot{\varphi}_1 \sin(\varphi_1 - \varphi_2)(\dot{\varphi}_1 - \dot{\varphi}_2)].
 \end{aligned}$$

based on (5). Substituting these into (6) yields

$$0 = l_1^2\ddot{\varphi}_1(m_1 + m_2) - m_2l_1l_2\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) - m_2l_1l_2\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2)(\dot{\varphi}_1 - \dot{\varphi}_2) +$$

$$m_2 \dot{\varphi}_1 \dot{\varphi}_2 l_1 l_2 \sin(\varphi_1 - \varphi_2) + gl_1(m_1 + m_2) \sin(\varphi_1).$$

Expanding $(\dot{\varphi}_1 - \dot{\varphi}_2)$ yields

$$0 = l_1^2 \ddot{\varphi}_1 (m_1 + m_2) + m_2 l_1 l_2 \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + m_2 l_1 l_2 \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) + gl_1(m_1 + m_2) \sin(\varphi_1).$$

Dividing by $l_1^2(m_1 + m_2)$ results in

$$0 = \ddot{\varphi}_1 + \frac{m_2}{m_1 + m_2} \frac{l_2}{l_1} \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \frac{m_2}{m_1 + m_2} \frac{l_2}{l_1} \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) + \frac{g}{l_1} \sin(\varphi_1)$$

which can be further simplified as

$$0 = \ddot{\varphi}_1 + \frac{m_2}{m_1 + m_2} \frac{l_2}{l_1} [\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2)] + \frac{g}{l_1} \sin(\varphi_1). \quad (8)$$

Proceeding analogously, (7) yields

$$0 = \ddot{\varphi}_2 + \frac{l_1}{l_2} [\ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) - \dot{\varphi}_1^2 \sin(\varphi_1 - \varphi_2)] + \frac{g}{l_2} \sin(\varphi_2) \quad (9)$$

after some rearrangements.

To simplify things further it will be assumed that $m_1 = m_2 = 1$ and $l_1 = l_2 = 1$ in arbitrary units yielding the following two equations of motion of the double pendulum based on (8) and (9):

$$\ddot{\varphi}_1 = -\frac{1}{2} [\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) + 2g \sin(\varphi_1)] \quad (10)$$

$$\ddot{\varphi}_2 = -[\ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) - \dot{\varphi}_1^2 \sin(\varphi_1 - \varphi_2) + g \sin(\varphi_2)] \quad (11)$$

2 Implementation

The implementation of equations (10) and (11) is straightforward as shown in figures 2 and 3. Both circuits require the functions $\sin(\varphi_1 - \varphi_2)$, $\cos(\varphi_1 - \varphi_2)$, $\sin(\varphi_1)$, and $\sin(\varphi_2)$, which are generated by three distinct sub-circuits such as the one shown

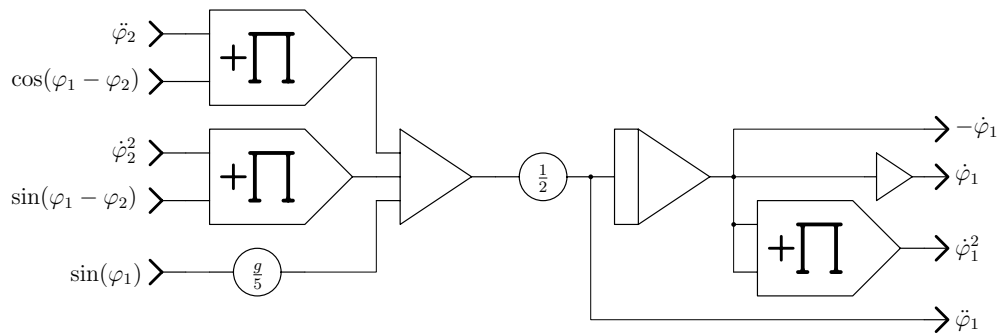


Figure 2: Implementation of equation (10)

in figure 4. These circuits yield sine and cosine of an angle based on the first time-derivative of this angle, so the simulation is not limited to a finite range for the two angles φ_1 and φ_2 .

It should be noted that each of the two integrators of each of these three harmonic function generators has a potentiometer connected to its respective initial value input. When a simulation run with given initial values for $\varphi_1(0)$ and $\varphi_2(0)$ is to be started, these potentiometers have to be set to $\cos(\varphi_1(0))$, $\sin(\varphi_1(0))$ and $\cos(\varphi_2(0))$, $\sin(\varphi_2(0))$, respectively.

Since $\dot{\varphi}_1$ and $\dot{\varphi}_2$ are readily available from the circuits shown in figures 2 and 3, two of these harmonic function generators can be directly fed with these values. The input for the third function generator is generated by a two-input summer as shown in figure 5.

With $\dot{\varphi}_1$ and $\dot{\varphi}_2$ and thus $\sin(\varphi_1)$, $\cos(\varphi_1)$, and $\sin(\varphi_2)$, $\cos(\varphi_2)$ readily available, the double pendulum can be displayed on an oscilloscope featuring two separate x, y -inputs by means of the circuit shown in figure 6. To yield a flicker-free display, this circuit requires a high-frequency input $\sin(\omega t)$ which can be obtained by two integrators in series with a summer, the output of which is fed back into the first integrators. This circuit is not shown here but can be found in [ULMANN 2017, p. 67].

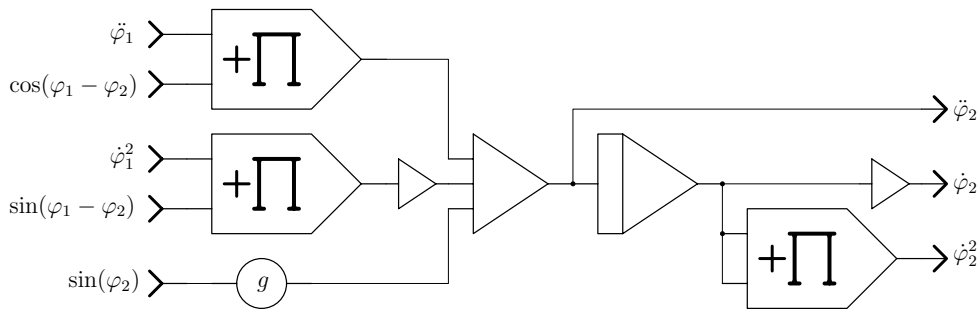


Figure 3: Implementation of equation (11)

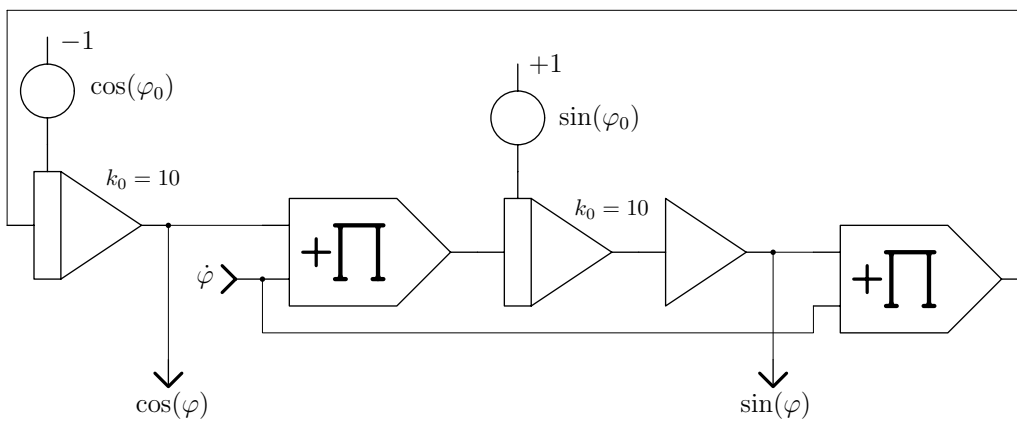


Figure 4: Generating $\sin(\varphi)$ and $\cos(\varphi)$

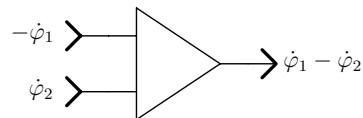


Figure 5: Computing $\dot{\varphi}_1 - \dot{\varphi}_2$

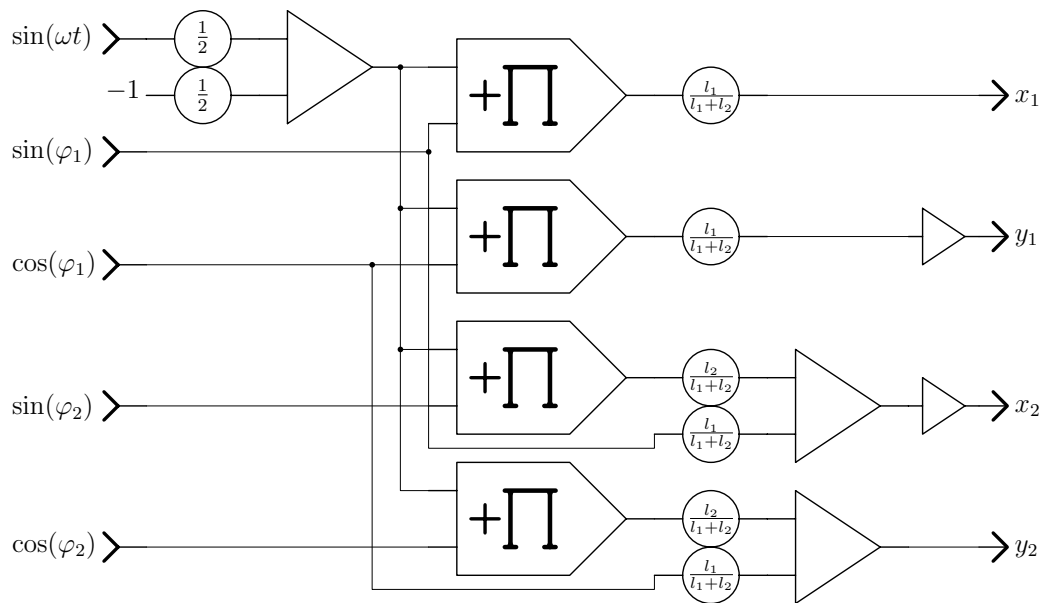


Figure 6: Display circuit for the double pendulum

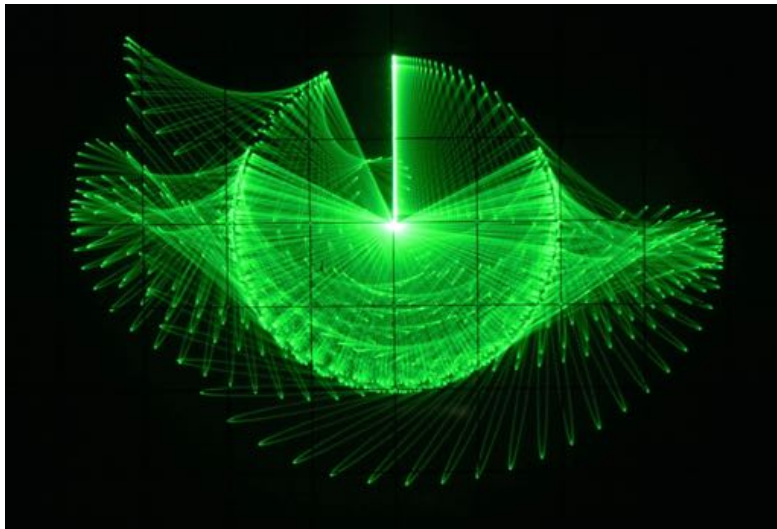


Figure 7: Long-term exposure of a double pendulum simulation run

3 Results

Figure 7 shows a long-term exposure of the movements of the double pendulum starting as an inverted pendulum with its first pendulum rod pointing upwards while the second one points downwards.

This simulation requires ten integrators, 15 summers, 16 multipliers, and 17 coefficient potentiometers.

References

[MAHREHOLTZ, 1968] OSKAR MAHREHOLTZ, *Analogrechnen in Maschinenbau und Mechanik*, BI Hochschultaschenb" ucher, 1968

[ULMANN 2017] BERND ULMANN, *Analog Computer Programming*, Create Space, 2017