The Hindmarsh-Rose-model of neuronal bursting

Beginning in the early 20th century, the behaviour of neurons has been described by increasingly realistic mathematical models. The very first of these models is called *integrate-and-fire* and is due to Lous Lapicque who developed it in 1907. A better model was developed in the early 1960s by Richard FitzHugh and J. Nagumo and is described by

$$\dot{v} = v - \frac{v^3}{3} - w + I_{\text{ext}} \text{ and}$$
$$\tau\dot{\omega} = v + a + -bw.$$

This model is still pretty simple and it can be shown that is basically equivalent to the van der Pol equation

$$\ddot{y} + \mu\left(y^2 - 1\right)\dot{y} + y = 0,$$

which was devised by Balthasar van der Pol in 1920 as a result of his pioneering work on vacuum tubes and oscillator circuits. Accordingly, the FitzHugh-Nagumo model is basically a relaxation oscillator[1] controlled by an external stimulus I_{ext}.

A much more interesting model is due to Hindmarsh and Rose[2] and consists of three coupled differential equations

$$\dot{x} = -ax^3 + bx^2 + y - z + I_{\text{ext}} \tag{1}$$
$$\dot{y} = -dx^2 + c - y \tag{2}$$
$$\dot{z} = r(s(x - x_r) - z) \tag{3}$$

with the parameters $a = 1$, $b = 3$, $c = 1$, $d = 5$, $r = 10^{-3}$, $s = 4$, $x_r = -\frac{8}{5}$ and initial conditions of 2 for all three integrators in the final setup.[3]

[1] In contrast to a harmonic oscillator which is typically based on an amplifier with suitable feedback, running in resonance mode, a relaxation oscillator switches abruptly between charge and discharge mode and thus yields non-harmonic output signals.

[2] See [Hindmarsh et al. 1982] and [Hindmarsh et al. 1984].

[3] See [Izhikevich 2010] for a thorough introduction to dynamical systems in neuroscience.

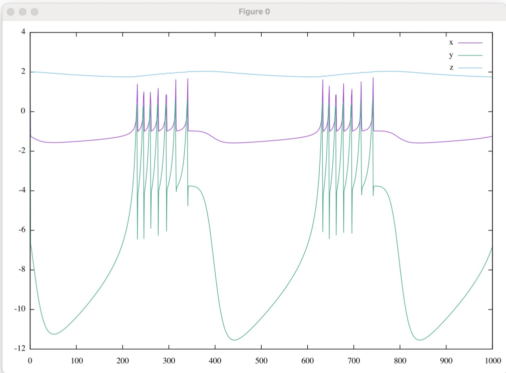

Figure 1: Unscaled Hindmarsh-Rose model

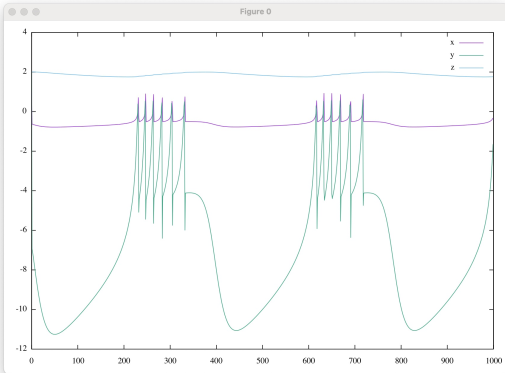

Figure 2: After first scaling step

A quick numeric integration yields the solution shown in figure 1. Obviously, the system must be scaled in order to be implemented on an analog computer. The variables x, y, and z all exceed the standard machine unit interval of $[-1, 1]$ by far.

First, we scale x by $\lambda_x = \frac{1}{2}$:[4]

$$\begin{aligned}
\dot{x} &= \frac{1}{2}\left(-8ax^3 + 4bx^2 + y - z + I_{\text{ext}}\right) \\
&= -4ax^3 + 2bx^2 + \frac{y}{2} - \frac{z}{2} + I_{\text{ext}} \\
\dot{y} &= -4dx^2 + c - y \\
\dot{z} &= r(s(2x - x_r) - z)
\end{aligned}$$

I_{ext} has also been scaled by λ_x and is now 1, which is very convenient. The behaviour of the system after this first scaling step is shown in figure 2. x is now well within $[-1, 1]$.

[4]Scaled variables are not denoted by a hat or star in the following as this would clutter the formulas considerable.

z can also be scaled with a factor $\lambda_z = \frac{1}{2}$ yielding the following system of equations:

$$\begin{aligned}
\dot{x} &= -4ax^3 + 2bx^2 + \frac{y}{2} - z + I_{\text{ext}} \\
\dot{y} &= -4dx^2 + c - y \\
\dot{z} &= \frac{1}{2}\left(r(s(2x - x_r) - 2z)\right) \\
&= r\left(s\left(x - \frac{x_r}{2}\right) - z\right)
\end{aligned}$$

The output of this system is shown in figure 3. All that is now left to do is scaling y with a "nice" value such as $\lambda_z = \frac{1}{15}$:

$$\begin{aligned}
\dot{x} &= -4ax^3 + 2bx^2 + 7.5y - z + I_{\text{ext}} \\
\dot{y} &= \frac{1}{15}\left(-4dx^2 + c - 15y\right) \\
&= -0.2664dx^2 + \frac{c}{15} - y \\
\dot{z} &= r\left(s\left(x - \frac{x_r}{2}\right) - z\right)
\end{aligned}$$

The initial condition of the y integrator is now so small that it can be safely set to 0.

Since coefficients on the LUCIDAC can be set in the range $[-10, 10]$, the parameters a, b, c, d, r, s, and x_r must not be scaled. $a = 1$ can be omitted altogether, and c can be replaced by its scaled value of $\lambda_y c = \lambda_y$. $b = 3$ and the inverse scaling factor $\frac{1}{\lambda_x}$ can be combined to a constant 6. The same can be done with the product $0.2664d \approx 1.332$.

Multiplying out the equation for $\dot{z}$ yields

$$\dot{z} = r\left(s\left(x - \frac{x_r}{2}\right) - z\right) = rsx - \frac{rsx_r}{2} - rz.$$

With all constants taken into account we get

$$\dot{z} = 0.004x + 0.0032 - \frac{z}{1000}$$

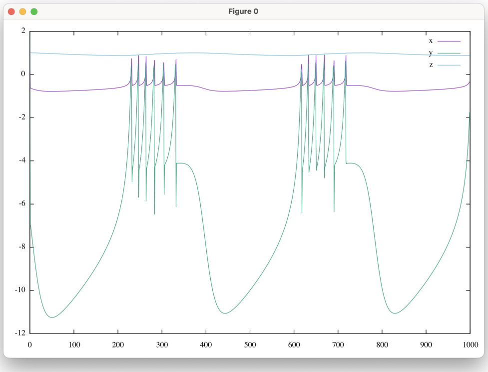

Figure 3: After second scaling step

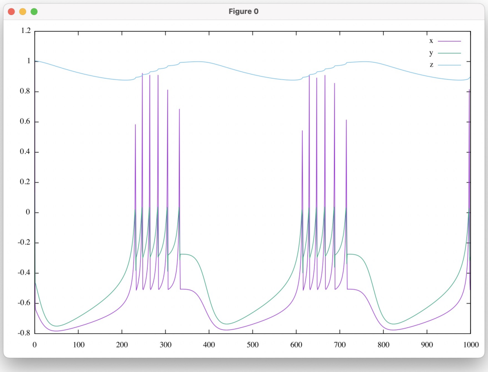

Figure 4: After third scaling step

yielding the following system of scaled equations:

$$\dot{x} = -4x^3 + 6x^2 + 7.5y - z + I_{\text{ext}}$$
$$\dot{y} = -1.332x^2 + 0.0666 - y$$
$$\dot{z} = 0.004x + 0.0032 - \frac{z}{1000}$$

Making use of different time scale factors of the integrators, the very small coefficients in the last equation can be scaled up to more convenient values. Without loss of generality, let $k_{0_x} = k_{0_y} = 10^4$ and $k_{0_z} = 10^2$,[5] yielding

$$\dot{x} = -4x^3 + 6x^2 + 7.5y - z + I_{\text{ext}} \qquad k_{0_x} = 10^4, \text{IC} = 1$$
$$\dot{y} = -1.332x^2 + 0.0666 - y \qquad k_{0_y} = 10^4, \text{IC} = 0$$
$$\dot{z} = 0.4x + 0.32 - 0.1z \qquad k_{0_z} = 10^2, \text{IC} = 1$$

These equations can now be setup on an analog or hybrid computer such as THE ANALOG THING, a Model-1, or a LUCIDAC as shown in figure 5.

[5] The important thing here is that the two k_0 values differ by a factor of 100, so on a that $k_{0_x} = k_{0_y} = 10^3$ (fast) and $k_{0_z} = 10$ (slow) would work as well.

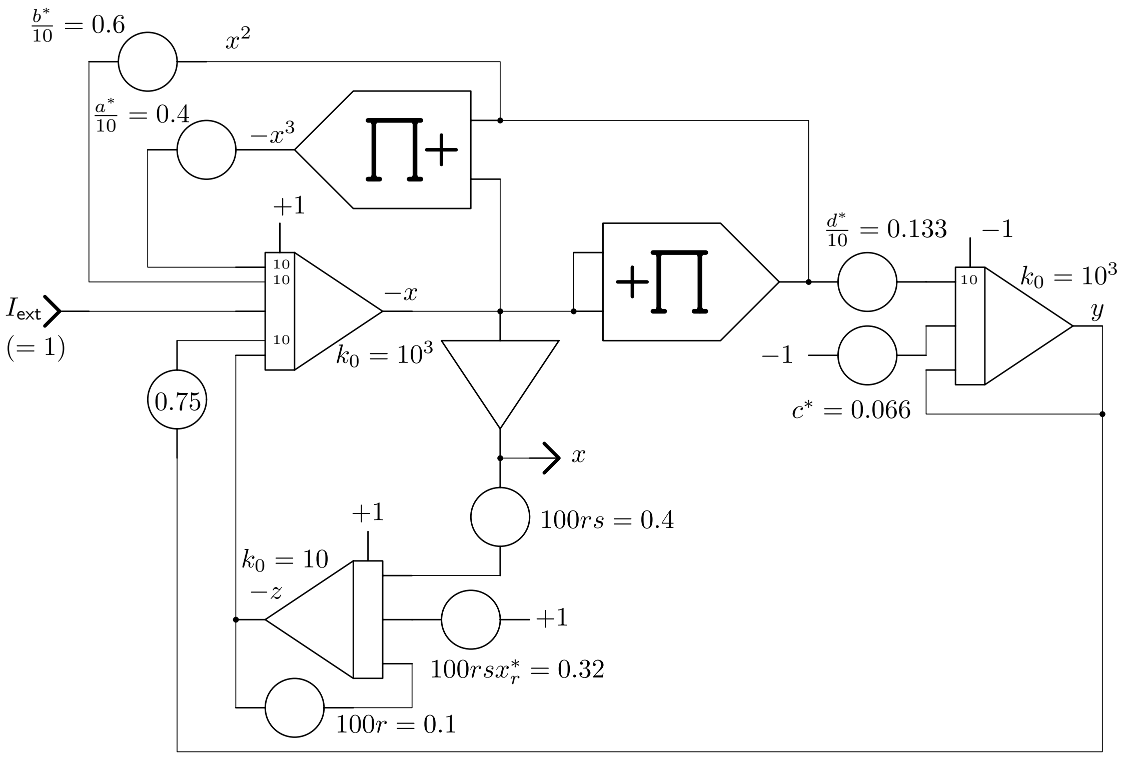

Figure 5: Scaled analog computer setup for the HINDMARSH-ROSE model

Figure 6: Typical result of spiking neuron simulation

Figure 6 shows a typical result obtained by this analog computer circuit with $I_{\text{ext}} = 1$. The yellow trace depicts the output potential of the neuron (x), the green trace corresponds to the amount of potassium channels (y), and the orange channel shows z.

References

[Hindmarsh et al. 1982] J. L. Hindmarsh, R. M. Rose, “A model of the nerve impulse using two first-order differential eequations”, in *Nature*, Vol. 296, 11th March 1982, pp. 162–164

[Hindmarsh et al. 1984] J. L. Hindmarsh, R. M. Rose, “A model of neuronal bursting using three coupled first order differential equations”, in *Prov. R. Soc. Lond.*, B 221, 87–102 (1984)

[Izhikevich 2010] Eugene M. Izhikevich, *Dynamical Systems in Neuroscience – The Geometry of Excitability and Bursting*, MIT Press, 1st paperback edition, 2010