Solving Legendre's differential equation

LEGENDRE's[1] differential equation has the form

$$(1-t^2)\ddot{y} - 2t\dot{y} + n(n+1)y = 0 \tag{1}$$

with the general solution

$$y = \alpha P_n(t) + \beta Q_n(t), \tag{2}$$

where $P_n(t)$ and $Q_n(t)$ are linearly independent and are called LEGENDRE functions of the 1st- and 2nd-kind. $P_n(t)$ are polynomials which is what we are interested in.

Due to the factor $1-t^2$ this equation can not be directly transformed into an analog computer setup as this would require a division, the denominator of which tends to 0. Instead equation (1) is rewritten as

$$\ddot{y} - t^2\ddot{y} - 2t\dot{y} + n(n+1)y = 0$$

which is then solved for $\ddot{y}$, yielding

$$\ddot{y} = t^2\ddot{y} + 2t\dot{y} - n(n+1)y. \tag{3}$$

This is insofar unusual as $\ddot{y}$ is not separated on the left side but also occurs on the right. In addition to that the terms $t^2\ddot{y}$ and $2t\dot{y}$ would require three multipliers if implemented in a naive way. One multiplier can be saved by rewriting equation (3):

$$\ddot{y} = t(t\ddot{y} + 2\dot{y}) - n(n+1)y.$$

To simplify notation, $n^+ = n(n+1)$ is introduced, finally yielding

$$\ddot{y} = t(t\ddot{y} + 2\dot{y}) - n^*y.$$

To select the desired solution $P_n(t)$ for this DEQ suitable initial conditions are required. Table 1 lists the $P_n(t)$ for $1 \le n \le 5$ with the resulting initial conditions for $\dot{y}(0)$ and $y(0)$. Obviously the problem must be scaled at least with respect to $\dot{y}$ as the

n	$P_n(t)$	$y(0)$	$\dot{y}(0)$
1	t	0	1
2	$\frac{3}{2}x^2 - \frac{1}{2}$	$-\frac{1}{2}$	0
3	$\frac{5}{2}x^3 - \frac{3}{2}x$	0	$-\frac{3}{2}$
4	$\frac{35}{8}x^4 - \frac{15}{4}x^3 + \frac{3}{8}$	$\frac{3}{8}$	0
5	$\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$	0	$\frac{15}{8}$

Table 1: The first five $P_n(t)$ solving LEGENDRE's DEQ with the resulting initial conditions

initial conditions nearly reach 2 for $n = 5$, thus giving rise to a scaling factor of $\frac{1}{2}$ for $\dot{y}$.

n^* must also be scaled – assuming a maximum value of $n = 5$ requires an additional scaling factor of $\frac{1}{n(n+1)} = \frac{1}{30}$. Some experimentation shows that $\ddot{y}$ quickly approaches 20 for $n = 5$ and $t \longrightarrow 1$. Taking all these scaling factors into account yields the program shown in figure 1,[2] the implementation of which on THE ANALOG THING is shown in figure 2.

To generate $P_5(t)$ with this setup, the initial conditions as well as n^* must be set according to table 1. It is $y(0) = 0$ and $\dot{y}(0) = \frac{15}{8}$. With an overall scaling factor of $\frac{1}{2\cdot 20}$ the initial value must be set to $\widehat{\dot{y}(0)} = \frac{15}{8\cdot 2\cdot 20} \approx 0.047$. $n = 5$ yields $n^* = 30$ so that the potentiometer following the integrator yielding y must be set to 0.05. The operation time is set to $\frac{1}{10}$s with the time scale factors shown.

Figure 3 shows function $y(t) = P_5(t)$ resulting from these values.[3] Experimenting with different initial conditions for $\dot{y}(0)$ and $y(0)$ yields results of the general form of equation 2.

[1] ADRIEN-MARIE LEGENDRE, 18.09.1752–09.01.1833

[2] The hat over the initial conditions denotes their scaled values.

[3] This is an x, y-plot with $x = t$ showing the flyback when THE ANALOG THING switched from OP to IC mode.

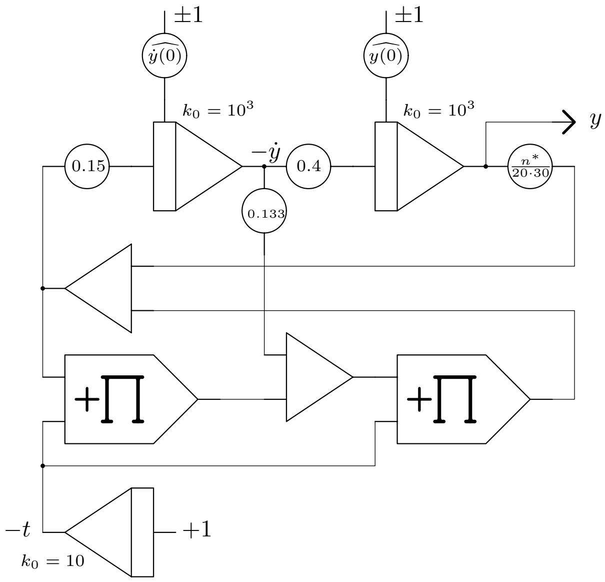

Figure 1: Analog computer program for solving the LEGENDRE DEQ

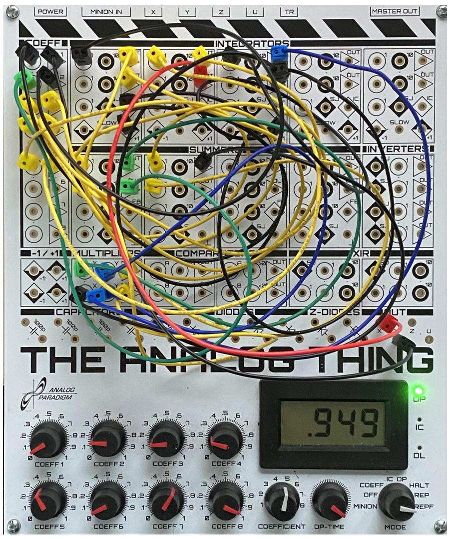

Figure 2: Actual setup of the problem on THE ANALOG THING

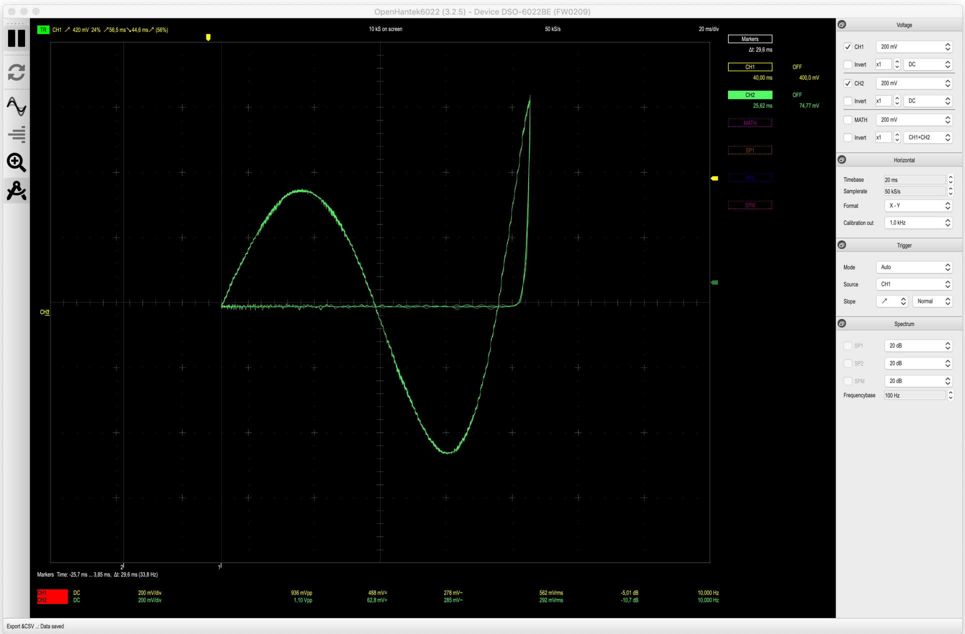

Figure 3: $P_5(t)$ with $0 \leq t \leq 1$