The logistic map

1 Introduction

A great account of the rather famous *logistic difference equation* or *logistic map* can be found in [May 1976]. In fact, this equation was popularized to a large degree by this publication. The equation itseld is of the general form

$$x_{t+1} = f(x_t),$$

i. e., it is based on discrete time steps. The actual equation is

$$x_{t+1} = a x_t (1 - x_t) \tag{1}$$

with a single parameter a. As simple as it looks, it exhibits a suprisingly complex behavior and often serves as an introductory example for chaotic systems.[1] In fact, its behavior is so rich that [May 1976, p. 467] writes:

> "*I would there fore urge that people be introduced to, say, [this] equation early in their mathematical education. [. . .] Such study would greatly enrich the student's intuition about nonlinear systems. Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realised that simple nonlinear systems do not necessarily possess simple dynamical properties.*"

2 Implementation

Implementing such a difference equation on an analog computer requires some form of storage element.[2] One could, for example, use two integrators in sequence with individual mode control to implement a *sample-and-hold* (*SAH*) circuit. Another option would be to

[1] See [Strogatz 2015, pp. 360 ff.] or [Steeb et al. 1989, pp. 14 ff.].
[2] It still feels strange for the author to deal with a discrete time system. . .

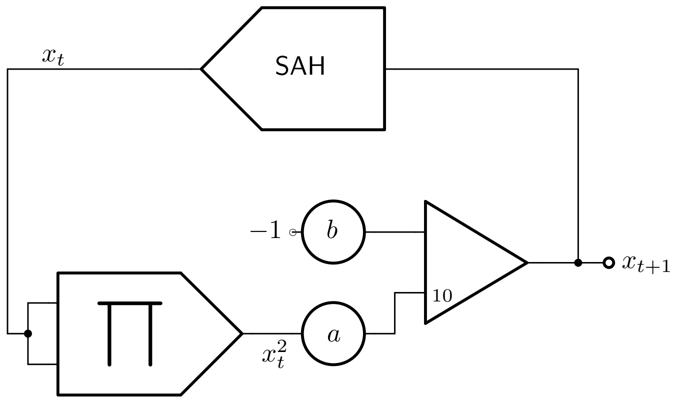

Figure 1: Implementation of equation (2equation.2)

implement a SAH element using operational amplifiers and capacitors for charge storage.[3] One might also use an *analog-digital-converter* (*ADC*), a digital latch, and a *digital-analog-converter* (*DAC*) to implement such a storage element which has been used in the following.[4]

The only problem with equation (1equation.1) is that $x_0 \neq 0$ is required as an initial condition, otherwise the system will just be stuck at zero. [Wu et al. 2022] propose a slightly different model exhibiting similar dynamics while avoiding this initial value problem:

$$x_{t+1} = b - ax_t^2 \tag{2}$$

This equation can be implemented directly on an analog computer such as THE ANALOG THING[5] as shown in figure 1figure.1 given a suitable delay element to introduce discrete time.

[3] See [National 1982] and [National 1992].

[4] Of course, a purely analog approach without the need for two conversion steps and a limited resolution as dictated by the ADC and DAC is preferable.

[5] See `https://the-analog-thing.org`, retrieved 25.10.2025.

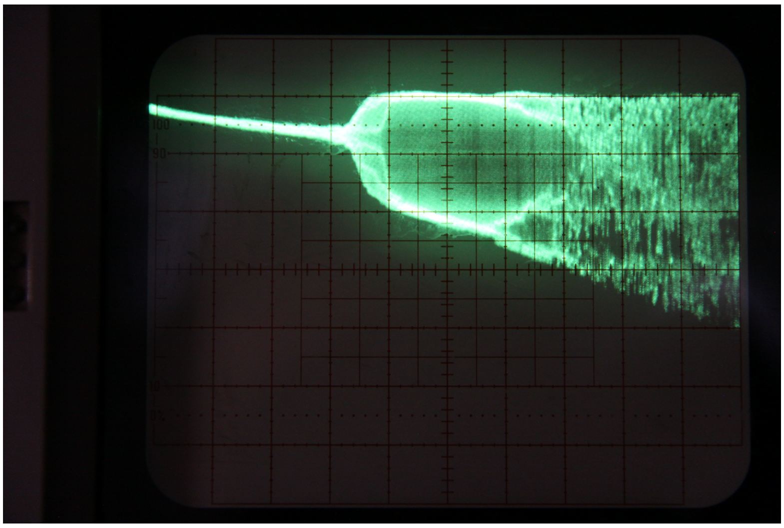

Figure 2: Typical bifurcation diagram for equation (2equation.2)

3 Results

Using this setup and replacing the parameter a with a multiplier fed by an integrator creating a linear ramp from 0 to 1 for automatic parameter variation it is possible to create a bifurcation diagram directly on an oscilloscope as shown in figure 2figure.2. To obtain this particular picture, the analog computer was first run in repetitive mode while the parameter b was manually adjusted so that the summer just did not go into overload. The picture then was taken from the screen of an analog storage oscilloscope.

Happy analog computing!

References

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