Newton’s Law of Cooling: An Introduction to Scaling

1 Introduction

This simulation applies scaling to solve a relatively simple differential equation involving Newton’s law of cooling. Initially at 100°C, a hot cup of tea is placed in a room that is maintained at an ambient constant temperature of 20°C.

For comparison, analytical, numerical, and analog computer solutions are displayed in table 1 on page 6.

Please note: Originally, the rocket equation derivation was going to be part of Application Note #9. Instead, it will be included in Application Note #10. Sorry for any inconvenience this may have caused.

2 Mathematical modeling

Starting with Newton’s law of cooling,

\[ \frac{dT}{dt} = -k(T - T_A) \text{ with } T(0) = T_0, \text{ where} \]

- \( T \) = temperature of the tea
- \( t \) = time
- \( k \) = temperature coefficient
- \( T_A \) = ambient temperature (assumed constant)
- \( T_0 \) = initial temperature of the tea

Letting \( k = 0.25 \text{ min}^{-1}, T_A = 20^\circ \text{C}, \) and \( T(0) = 100^\circ \text{C}, \)

\[ \frac{dT}{dt} = -0.25\text{min}^{-1}(T - 20^\circ \text{C}) \text{ with } T(0) = 100^\circ \text{C} \] (2)

Following a bit of calculus (details will be provided for the scaled version),
\[ T(t) = 80^\circ C e^{(-0.25 \text{ min}^{-1}t)} + 20^\circ C \quad (3) \]

As a reminder, operational amplifier (op amp) input/output voltages are between \( V_{EE} \) and \( V_{CC} \). For this project, 9-Volt batteries were used \( (V_{EE} = -9 \text{ Volts and } V_{CC} = +9 \text{ Volts}) \). Allowing for a safety of margin, to prevent saturation, voltages are kept between -6 Volts and +6 Volts.

The issue of magnitudes is obvious. A value like 100\(^\circ\)C is just too big for direct conversion to 100 Volts. So, a temperature of 100\(^\circ\)C will be reduced (scaled down) to an analog voltage of 6 Volts and 20\(^\circ\)C will be reduced to an analog voltage of 1.20 Volts.

Time will be scaled such that 1 minute = 1 second. Easy enough!

To start, let

\[ R \equiv \text{Reducing scale factor} = \frac{T_{\text{max}}}{V_{\text{max}}} = \frac{100^\circ C}{6 \text{ Volts}} = \frac{50^\circ C}{3 \text{ Volts}}. \]

In general, \( R = \frac{T}{V} \) or \( T = RV \).

Replacing \( T \) with \( RV \), (1) becomes

\[
\frac{d(RV)}{dt} = -k(RV - T_A) \quad \text{with } V(0) = \frac{T(0)}{R} = \frac{T_0}{R} \\
R \frac{dV}{dt} = -kR(V - T_A/R) \quad \text{with } V(0) = \frac{T(0)}{R} = \frac{T_0}{R} \\
dV/dt = -k(V - V_A) \quad \text{with } V_A = \frac{T_A}{R} \text{ and } V(0) = \frac{T(0)}{R} = \frac{T_0}{R}
\]

Inserting values, but omitting units for clarity,

\[
dV/\left(V - 1.20\right) = -0.25 \quad \text{with } V(0) = 100/(50/3) \\
dV/\left(V - 1.20\right) = -0.25(V - 20/(50/3)) \quad V(0) = 100/(50/3) \quad (4) \\
dV/\left(V - 1.20\right) = -0.25(V - 1.20) \quad \text{with } V(0) = 6.00
\]

\[
\int_{6}^{T} \frac{dV}{V - 1.20} = -0.25 \int_{0}^{t} dt
\]
Integrating by inspection, and noting that \( V > 1.20 \),

\[
\ln((V - 1.20)/(6 - 1.20)) = -0.25t
\]

\[
\ln((V - 1.20)/4.8) = -0.25t
\]

\[
(V - 1.20)/4.8 = e^{-0.25t}
\]

\[
V = 4.80 e^{-0.25t} + 1.20
\]

Inserting units,

\[
V(t) = 4.80 \text{ Volts } e^{-0.25 \text{ min}^{-1}t} + 1.20 \text{ Volts} \tag{5}
\]

Since \( T = RV \),

\[
T(t) = 50^\circ/3 \text{ Volts } \times [4.80 \text{ Volts } e^{-0.25 \text{ min}^{-1}t} + 1.20 \text{ Volts}]
\]

\[
T(t) = 80^\circ \text{C } e^{-0.25 \text{ min}^{-1}t} + 20^\circ \text{C}, \text{ which is identical to (3)}
\]

3a Computer setup (scaled, patch cord version)

![Diagram of computer setup](image)

Figure 1: Computer setup for Newton's law of cooling
3b  Computer setup (op amp/discrete component version)

Operational amplifier type:
μA741CN

Arrow Color Key:
Red arrows connect to VCC.
Blue arrows connect to VEEL.
Black arrows connect to ground.
Green arrows connect to component indicated.

Figure 2: Basic breadboard layout
4 Numerical Method

(Modified Euler method using a hand-held programmable calculator)

Code: TI-BASIC

PROGRAM:COOLING
:ClrHome:ClrDraw
:”NEWTONS LAW”
:”OF COOLING”
:”DV/DT=-0.25(V-1.20)”
:”WITH V(0)=6.00”
:”V=VOLTAGE ANALOG”
:”OF TEMPERATURE”
:”T =TIME”
:”PARAMETERS:”
:0→T:6.00→V:0.25→H
:”H=STEP SIZE”
:Fix 1
:Lbl 1
:If T>20:Then
:Goto 2: Else
:Disp {T,V}
:-0.25(V−1.20)→F
:V+HF→W
:T+H→T
:-0.25(W−1.20)→S
:(F+S)/2→A
:V+AH→V
:Pause
:Goto 1
:Lbl 2
:End
5 Results

Figure 3: Voltage vs time simulation*

*For this application note, the oscilloscope display was produced during a single run using a differential equation analog computer constructed from operational amplifiers and discrete components with tolerances within 10%.

<table>
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<tr>
<th>t (s) scaled from minutes</th>
<th>Analog Computer V (Volts) Estimated from oscilloscope</th>
<th>Analog Computer Converted T (°C)</th>
<th>Analytical T (°C)</th>
<th>Numerical T (°C)</th>
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Table 1: Solution Comparisons
Figure 4: Differential Equation Analog Computer

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